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ABSTRACT

This paper examines the use of the extended Kalman filter for estimating various quantities in typical interceptor-target engagements. The extended Kalman filter is used to estimate the relative position of the target, the relative velocity of the target and the vector perpendicular to the target velocity. The target is assumed to be non accelerating. The target is observed via range and bearing measurements and it is assumed that the interceptor's own velocities are known.

The performance and stability of the extended Kalman filter are examined under a variety of initialisation errors, engagement configurations, and measurement noise variances. Simulation studies demonstrate that the required quantities can be estimated but that the performance of the filter is dependent on the configuration of the engagement.

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Filtering for Precision Guidance: The Extended Kalman Filter

EXECUTIVE SUMMARY

This paper examines the use of the extended Kalman filter for estimating various quantities in a typical interceptor-target engagement. The quantities estimated are required for the purposes of precision guidance. New tactical demands of missiles require advanced guidance and control methodologies. These new control methodologies require more information about the target than is available using existing target tracking algorithms. In the simplest terms, new precision guidance control objectives require knowledge of the missile impact angle on the target in addition to the more traditional information about the target's relative position and velocity. This new requirement makes re-design and re-evaluation of the extended Kalman filter necessary.

In a typical interceptor-target engagement, the information required to successfully control the interceptor needs to be estimated from radar and inertial measurement unit (IMU) observations. The extended Kalman filter is a generic algorithm that can be used in this situation to estimate the required information. Unfortunately, this target tracking estimation problem is notoriously difficult and the performance and stability of the extended Kalman filter needs to be examined extensively prior to use.

In this paper the extended Kalman filter is applied to the interceptor-target problem and its performance is examined via simulation studies in a variety of situations. An understanding of the performance of the extended Kalman filter for target tracking will enable a more thorough and efficient response to Australian Defence Force requirements for assessment, evaluation, advice and modification of weapon systems.

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Jason Ford joined the Guidance and Control group in the Weapons Systems Division in February 1998. He received B.Sc. (majoring in Mathematics) and B.E. degrees from the Australian National University in 1995. He also holds the PhD (1998) degree from the Australian National University. His thesis presents several new on-line locally and globally convergent parameter estimation algorithms for hidden Markov models (HMMs) and linear systems.

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His interests include: weapon navigation technologies, inertial testing, embedded systems, hardware-in-the-loop testing of systems and control hardware systems.

Contents

Notation

$x_t^T, y_t^T, u_t^T, v_t^T, a_t^T$ and b_t^T	x-y components of target's position, velocity and acceleration, respectively.
$x_t^I, y_t^I, u_t^I, v_t^I, a_t^I$ and b_t^I	x-y components of interceptor's position, velocity and acceleration, respectively.
x_t, y_t, u_t and v_t	x-y components of relative position and velocity, respectively.
z_k	measurements.
$(\bar{u}_t, \bar{v}_t)$	vector perpendicular to (u_t, v_t) with magnitude (u_t^I, v_t^I).
V_T and V_I	magnitude of the target and interceptor's velocities, respectively.
α	the ratio of V_I and V_T.
X_t and X_k	the relative state in continuous and discrete time, respectively.
θ_t^T and θ_t^S	the target's heading angle and the bearing to the target, respectively.
$\omega_t, \omega_t^{T,long}$ and $\omega_t^{T,lat}$	noises used to model target accelerations.
w_k^R, w_k^θ, w_k and n_k	model observation noises.
A, B, C and G	system matrices.
$a_k(.), b_k(.)$ and $c_k(.)$	non-linear system functions.
A_k, B_k and C_k	linearisation matrices.
Q_k, R_k, Q_k^* and R_k^*	noise and pseudo noise covariance matrices, respectively.
$\bar{X}_{k\|k-1}$ and $\bar{X}_{k-1\|k-1}$	extended Kalman filter state estimates.
$\bar{P}_{k\|k-1}$ and $\bar{P}_{k-1\|k-1}$	extended Kalman filter covariance estimates.
h	sampling period.

1 Introduction

Precision guidance of weapon systems is a computationally and conceptually demanding problem. In the past, linear control methods have been applied with some success to design controllers that minimise the miss distance between the target and the interceptor. The advances in computational hardware and changes in defence requirements have resulted in an increased need (or desire) for more precise performance from weapon systems. Recently, a precision guidance problem has been proposed that involves ensuring not only the miss distance is minimised but also that the angle of impact is controlled to a desired impact angle. Controlling an interceptor to a desired impact angle requires access to more information than is needed by traditional minimum miss-distance controllers. This paper examines the use of the extended Kalman filter to provide the information required by the precision guidance law proposed in [2].

The most famous and commonly used assumptions are that the system is Gauss-linear and that the noises are Gaussian. Under these system assumptions, the Kalman filter is the optimal filter for estimating system states [1]. Because the Kalman filter is optimal (in a minimum mean square sense) and finite dimensional (the probability density function can be represented by a finite number of moments), it has been applied to a large variety of filtering problems. The continuing success of the Kalman filter in many applications has encouraged its use even in situations where the underlying system is clearly non-linear.

Apart from the Kalman filter, there are very few finite-dimensional optimal filters for stochastic filtering problems. For general non-linear problems, when an finite-dimensional optimal filter is not possible, sub-optimal numeric or approximate approaches must be used. The simplest approach is to linearise the non-linear model about various operating points and perform filtering with the extended Kalman filter (a generalisation of the Kalman filter). Details of the extended Kalman filter (EKF) are given in this paper as well as some recent stability results that establish under which conditions the extended Kalman filter will provide a reasonable filtering solution.

In a typical interceptor-target engagement, measurements of the target are relative range and bearing to the target. These measurements are non-linearly related to the relative position and velocity of the target in cartesian co-ordinates. The Kalman filter is not

appropriate for this filtering problem because of the non-linearity in the measurement process (even if the state dynamics of the target are approximated as being Gauss-linear). However, after appropriate linearisation, the extended Kalman filter can be applied to the target tracking state estimation problem.

The target tracking problem in this paper is more difficult than tracking problems previously solved using an extended Kalman filter because the precision guidance control problem requires information in addition to the usual target location and velocity information. To enable estimation of this additional information, we assume that radar measurements (ie. relative range and bearing) and the interceptor velocity in an absolute reference frame are measured. We show that measurement of the interceptor velocity is required to ensure the desired target state information is observable.

The stability and performance of the extended Kalman filter is known to vary significantly for different state estimation problems. We examine the stability and performance of the filter through both theoretical results and simulation studies. The error performance of the filter is also evaluated for a variety of initialisation conditions, engagement configurations, and measurement noise variances.

The paper is organised as follows: In Section 2, the target tracking problem considered in this paper is presented. Both continuous time and discrete time equations are given to describe the dynamics and the measurement process of the engagement. A typical engagement configuration is presented to demonstrate a likely encounter. Then, observability of the filtering problem is discussed. In Section 3, the extended Kalman filter is presented and applied to the target tracking problem presented in the previous section. Stochastic stability results for the extended Kalman filter are then presented and applied to the target tracking problem in a typical engagement. In Section 4, some implementation issues are discussed before simulation studies are presented in Section 5. These simulation studies examine the stability and error performance of the extended Kalman filter. Finally, some conclusions are given in Section 6.

2 Application: Target Tracking

In this section we present the target tracking problem which is a companion to the precision guidance control problem considered in [2]. For simplicity, at least partially justified by the separation principle [12], we separate the tracking problem from the control design problem, and assume that no control action is performed. The results presented in this paper can be applied to the control problem by adding a measured input signal for the interceptor dynamics.

The terminal phase of a interceptor-target engagement is described below.

For simplicity consider an engagement defined in continuous time and let the following definitions be in a 2-D Euclidean frame. Let (x_t^I, y_t^I) and (x_t^T, y_t^T) be the position of the interceptor and target respectively where the subscript $t \geq t_0$ denotes time. Then let (u_t^I, v_t^I), (u_t^T, v_t^T), (a_t^I, b_t^I) and (a_t^T, b_t^T) be the velocity and acceleration of the interceptor and target respectively. Also define $(\bar{u}_t, \bar{v}_t)$ to be the vector perpendicular to (u_t^T, v_t^T) with the same magnitude as (u_t^I, v_t^I). To create this perpendicular vector define the following:

$$\begin{aligned} \alpha &:= \frac{V_I}{V_T}, \quad \gamma := \frac{\frac{d\alpha}{dt}}{(1+\alpha^2)} \quad \text{and} \\ \beta &:= \gamma\alpha \end{aligned} \tag{2.1}$$

where $V_I := \sqrt{(u_t^I)^2 + (v_t^I)^2}$ and $V_T := \sqrt{(u_t^T)^2 + (v_t^T)^2}$.

Observations of the engagement are commonly related to the relative dynamics of the interceptor and target so we introduce the following state variable, $X_t := [x_t, y_t, u_t, v_t, \bar{u}_t, \bar{v}_t, a_t^T, b_t^T]'$, where $x_t := x_t^T - x_t^I$ etc. and $'$ denotes the transpose operation.

The dynamics of $(\bar{u}_t, \bar{v}_t)$ are given in [2], hence the dynamics of the state can be expressed as follows:

$$\begin{aligned} \frac{dX_t}{dt} &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \beta & -\gamma & 1 & -\alpha \\ 0 & 0 & 0 & 0 & \gamma & \beta & \alpha & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} X_t + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_t^I \\ b_t^I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \cos\theta_t^T & -\sin\theta_t^T \\ \sin\theta_t^T & \cos\theta_t^T \end{bmatrix} \begin{bmatrix} \omega_t^{T,long} \\ \omega_t^{T,lat} \end{bmatrix} \\ \frac{dX_t}{dt} &= AX_t + Bu_t + G(X_t)\omega_t \end{aligned} \tag{2.2}$$

where $\theta_t^T = \tan^{-1}(v_t^T/u_t^T)$ is the target heading angle, $u_t := [a_t^I, b_t^I]'$, and $\omega_t := [\omega_t^{T,long}, \omega_t^{T,lat}]$. Although target acceleration is deterministically controlled by the target, in this model the target acceleration has been approximated by a "jinking" type model through the noises $\omega^{T,long}$ and $\omega^{T,lat}$. This acceleration model is simplistic (see [9] for more realistic target models) but is a reasonable interceptor-target model in many situations.

Assume that the state is observed at evenly spaced distinct time instants $t_0, t_1, \ldots, t_k, \ldots$. Let index k denote the kth observation corresponding to the time instant $t = t_k$. Consider the following observation process

$$z_k = f(X_{t_k}, w_k^R, w_k^\theta) = \begin{bmatrix} R_k & + & R_k w_k^R \\ \theta_k^S & + & w_k^\theta \\ & u_{t_k}^I & \\ & v_{t_k}^I & \end{bmatrix} \tag{2.3}$$

where $R_k = \sqrt{x_{t_k}^2 + y_{t_k}^2}$, $\theta_k^S = \tan^{-1}(y_{t_k}/x_{t_k})$, w_k^R,w_k^θ are uncorrelated zero-mean Gaussian noises with variances σ_R^2 and σ_θ^2 respectively. Note that the observation noise on range measurements is assumed to be range dependent. Also, the interceptor velocities can be written as follows (see [2]):

$$\begin{aligned} u_{t_k}^I &= \frac{1}{1+\alpha^2}(\bar{u}_{t_k} + \alpha \bar{v}_{t_k}) - u_{t_k} \\ v_{t_k}^I &= \frac{1}{1+\alpha^2}(\bar{v}_{t_k} - \alpha \bar{u}_{t_k}) - v_{t_k} \end{aligned} \tag{2.4}$$

It is useful to consider a discrete-time representation of the continuous-time state equation (2.2) obtained through sampling theory. Let $h = t_k - t_{k-1}$, then using the sample hold approximation, the discrete-time state equation, for $k = 0, 1, \ldots$ is

$$\begin{aligned} X_{t_{k+1}} &= e^{Ah} X_{t_k} + G_{t_k} \bar{\omega}_{t_k} \quad \text{or} \\ X_{k+1} &= \bar{A} X_k + G_k \bar{\omega}_k \end{aligned} \tag{2.5}$$

where $\bar{\omega}_k = 1/h \int_{t_k}^{t_{k+1}} \omega_t dt$ and X_k denotes the discrete-time representation of X_t (likewise G_k etc.). The variance of $\bar{\omega}_k$ is $1/h$ times the variance of ω_t. The observation process for the discrete time model can be written as follows.

$$z_k = c_k(X_k) + n_k \tag{2.6}$$

where

$$c_k = \begin{bmatrix} R_k \\ \theta_k^S \\ u_{t_k}^I \\ v_{t_k}^I \end{bmatrix} \quad \text{and} \quad n_k = \begin{bmatrix} R_k w_k^R \\ w_k^\theta \\ 0 \\ 0 \end{bmatrix} \tag{2.7}$$

2.1 A Typical Engagement

A typical engagement is shown in Figure 1. The engagement commences at a distance of 5000 m. The interceptor is traveling at a velocity of 1000 ms^{-1} in a direction 36° (measured in a counter-clockwise direction) from the x-axis. The target is traveling at a velocity of 660 ms^{-1} in a direction of 120° from the x-axis. Hence, the initial conditions are $(x_0^I, y_0^I, u_0^I, v_0^I) = (0, 0, 1000\cos(36°), 1000\sin(36°))$ and $(x_0^T, y_0^T, u_0^T, v_0^T) = (5000, 0, 660\cos(120°), 660\sin(120°))$ where distances are in units of m and velocities are in units of ms^{-1}. Through out this paper near collision course geometries are considered and no control action is taken by the interceptor. The absence of control action allows a simpler analysis of filtering performance.

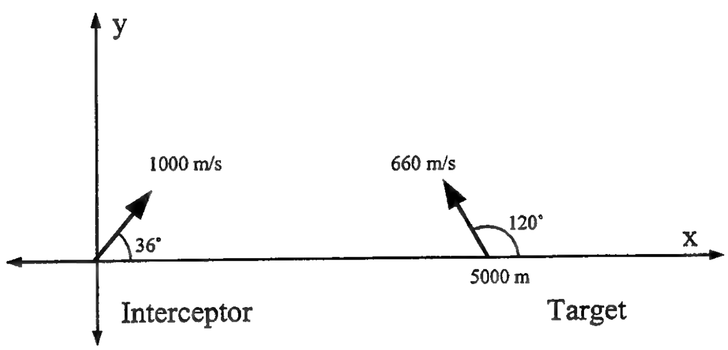

Figure 1 (U): Engagement configuration. The interceptor and target are roughly heading towards the same point (but collision does not occur).

2.2 Observability

Before proceeding to introduce the extended Kalman filtering solution to this problem we briefly examine the observability of the system defined in the previous section. System observability is necessary to ensure the state information needed for the precision guidance problem can be estimated from the measurements. Observability is defined as the ability to determine the value of the state from the system measurements. For linear systems, observability can be tested via the rank of the observability grammian which is defined as

follows:

$$G_O := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix} \tag{2.8}$$

where $C \in R^{(M\times N)}$ is the measurement output matrix and $A \in R^{(N\times N)}$ is the state transition matrix. Here N, M are the dimensions of the state and output processes respectively.

If G_O has rank N then the system is observable. For non-linear systems, observability of the system (in a local sense) at various time-instants can be examined via the linearised model, see [11] for details on observability of non-linear systems. It becomes clear when the observability grammian is examined for this problem that measurements of the interceptor velocity are necessary to ensure observability. The measurement equation (2.6) is a non-linear function of the state and linearisation at X_k gives

$$\begin{aligned} C_k &= \left.\frac{\partial c_k(X)}{\partial X}\right|_{X=X_k} \\ &= \begin{bmatrix} x_k/R_k & y_k/R_k & 0 & 0 \\ -y_k/R_k^2 & x_k/R_k^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1/(1+\alpha^2) & -\alpha/(1+\alpha^2) \\ 0 & 0 & \alpha/(1+\alpha^2) & 1/(1+\alpha^2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{2.9}$$

With the above C_k, the observability grammian for this problem has rank 8 (ie. is full rank). If the interceptor velocity measurements are not available then the grammian only has rank 6. This means that the full state vector can not be determined from measurements of only the range and bearing.

3 Extended Kalman Filter

Filtering for non-linear systems is a difficult problem for which few satisfactory solutions can be found. The sub-optimal approach considered in this section, that works in some situations, is based on an extension of the Kalman filter known as the extended Kalman filter.

The extended Kalman filter is posed by linearisation of a non-linear system. Consider the following non-linear system defined for k a non-negative integer:

$$\begin{aligned} X_{k+1} &= a_k(X_k) + b_k(X_k)w_k, \quad \in R^{(N\times 1)} \\ z_k &= c_k(X_k) + v_k, \qquad\qquad \in R^{(M\times 1)} \end{aligned} \tag{3.1}$$

where $a_k(.) \in R^{(N\times 1)}$, $b_k(.) \in R^{(N\times p)}$ and $c_k(.) \in R^{(M\times 1)}$ are non-linear functions of the state and $w_k \in R^{(p\times 1)}$, $v_k \in R^{(M\times 1)}$.

Let us define the following quantities:

$$A_k = \left.\frac{\partial a_k(X)}{\partial X}\right|_{X=\bar{X}_{k|k-1}}, \quad B_k = \left.\frac{\partial b_k(X)}{\partial X}\right|_{X=\bar{X}_{k|k-1}} \text{ and } \quad C_k = \left.\frac{\partial c_k(X)}{\partial X}\right|_{X=\bar{X}_{k|k-1}}. \tag{3.2}$$

Here $A_k \in R^{(N\times N)}$, $B_k \in R^{(N\times p)}$ and $C_k \in R^{(M\times N)}$.

Let us also introduce matrices Q_k^* and R_k^* which are related to the covariance matrices for noises w_k and v_k. However, as will be shown later in Section 3.1, the matrices Q_k^* and R_k^* need not equal the actual noise covariance matrices and in fact other positive definite matrices are often better choices.

The extended Kalman filter is implemented using the following equations:

$$\begin{aligned} \bar{X}_{k|k-1} &= a_{k-1}(\bar{X}_{k-1|k-1}) \\ \bar{P}_{k|k-1} &= A_{k-1}\bar{P}_{k-1|k-1}A'_{k-1} + B_{k-1}Q_k^*B'_{k-1} \\ K_k &= \bar{P}_{k|k-1}C'_k\left[C_k\bar{P}_{k|k-1}C'_k + R_k^*\right]^{-1} \\ \bar{X}_{k|k} &= \bar{X}_{k|k-1} + K_k\left[z_k - c_k(\bar{X}_{k|k-1})\right] \\ \bar{P}_{k|k} &= \bar{P}_{k|k-1} - K_kC_k\bar{P}_{k|k-1} \end{aligned} \tag{3.3}$$

These equations are not optimal or linear because A_k and C_k depend on $\bar{X}_{k|k-1}$. The symbols $\bar{X}_{k|k-1}$, $\bar{X}_{k-1|k-1}$, $\bar{P}_{k|k-1}$ and $\bar{P}_{k-1|k-1}$ loosely denote approximate conditional means and covariances respectively.

The extended Kalman filter presented above is based on first order linearisation of a linear system, but there are many variations on the extended Kalman filter based on second order linearisation or iterative techniques. Although the extended Kalman filter or other linearisation techniques are no longer optimal filters, these filters can provide reasonable filtering performance in some situations.

3.1 Stochastic Stability of the Extended Kalman Filter

A key question when applying an extended Kalman filter to a particular non-linear problem is when will the extended Kalman filter be stable and when will it diverge? Hueristic arguments have been used to suggest that if the non-linearities are "small" enough, and the filter is initialised well enough, then the filter should be stable (solid results are presented in Theorem 1). This hueristic argument has encouraged the use of the extended Kalman filter in a wide variety of signal processing, control and filtering problems. However, without any solid stability results, the error behaviour of the extended Kalman filter needs to be examined through testing whenever it is applied [6, 1].

Recently, solid stability results have established conditions on the non-linearities and initial conditions which ensure that the extended Kalman filter will produce estimates with bounded error [7, 8]. These results answer some of the stability questions surrounding the extended Kalman filter [7, 8]. This section repeats the stability results of [8].

Consider again the non-linear system (3.1) defined in Section 3:

$$\begin{aligned} X_{k+1} &= a_k(X_k) + b_k(X_k)w_k \\ z_k &= c_k(X_k) + v_k. \end{aligned} \tag{3.4}$$

Let us define the following quantities

$$\begin{aligned} \varphi(X_k, \bar{X}_k) &:= a_k(X_k) - a_k(\bar{X}_k) - A_k(X_k - \bar{X}_k) \\ \chi(X_k, \bar{X}_k) &:= c_k(X_k) - c_k(\bar{X}_k) - C_k(X_k - \bar{X}_k) \end{aligned}$$

where $\bar{X}_k$ is some estimate of the state (see Figure 2 for a graphic interpretation of $\varphi(X_k, \bar{X}_k)$). Also, define the estimation error as $\tilde{X}_{k|k} := X_k - \bar{X}_{k|k}$. Then the following theorem is presented in [8].

Theorem 1 *(Theorem 3.1) Consider the nonlinear system (3.1) and the extended Kalman filter presented in Section 3. Let the following hold:*

1. *There are positive real numbers* $\bar{a}, \bar{c}, \underline{p}, \bar{p}, \underline{q}, \underline{r} > 0$ *such that the following bounds hold for all* $k \geq 0$*:*

$$||A_k|| \leq \bar{a}$$

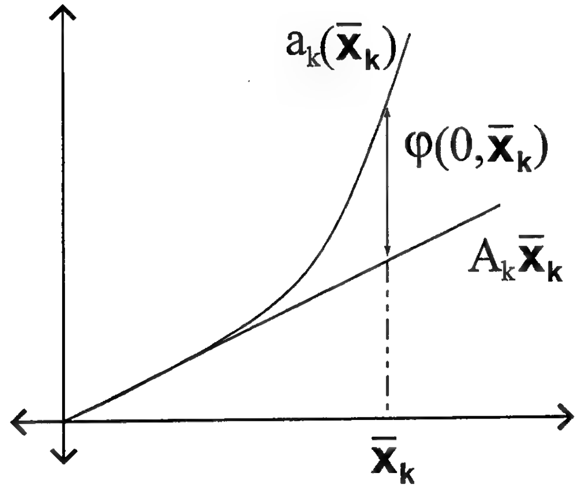

Figure 2 (U): Graphical interpretation of $\varphi(X_k, \bar{X}_k)$.

$$\begin{aligned} ||C_k|| &\leq \bar{c} \\ \underline{p}I &\leq P_{k|k-1} \leq \bar{p}I \\ \underline{q} &\leq Q_k^* \\ \underline{r} &\leq R_k^*. \end{aligned} \tag{3.5}$$

2. *A_k is nonsingular for all $k \geq 0$.*

3. *There are positive numbers $\epsilon_\varphi, \kappa_\varphi > 0$ such that:*

$$||\varphi(X_k, \bar{X}_k)|| \leq \kappa_\varphi ||X_k - \bar{X}_k||^2 \tag{3.6}$$

for X_k, $\bar{X}_k$ with $||X_k - \bar{X}_k|| \leq \epsilon_\varphi$ for all k, where $\bar{X}_k$ is any estimate of X_k at time k.

4. *There are positive numbers $\epsilon_\chi, \kappa_\chi > 0$ such that:*

$$||\chi(X_k, \bar{X}_k)|| \leq \kappa_\chi ||X_k - \bar{X}_k||^2 \tag{3.7}$$

for X_k, $\bar{X}_k$ with $||X_k - \bar{X}_k|| \leq \epsilon_\chi$ for all k, where $\bar{X}_k$ is any estimate of X_k at time k.

Then the estimation error is bounded with probability one, provided the initial estimation error satisfies

$$||\tilde{X}_o|| \leq \epsilon \tag{3.8}$$

and the covariances of the noise terms are bounded via $Q_k \leq \delta I$ and $R_k \leq \delta I$ for some δ, ϵ.

3.1.0.1 Remark

1. The proof of Theorem 1 is given in [8].
2. This result states that if the non-linearity is small then the EKF is stable if initialised close enough to the true initial value. The greater the deviation from linearity the better the initialisation needs to be.

The proof presented in [8] provides a technique for calculating conservative bounds for ϵ and δ. Although simulation studies suggest that ϵ and δ can be significantly larger than these bounds in some situations, these bounds can be useful in understanding the likely performance of a filter.

We define the following to repeat the bounds presented in [8]:

$$\begin{aligned} \bar{\epsilon} &:= \min(\epsilon_\varphi, \epsilon_\chi), & (3.9)\\ \bar{\kappa} &:= \kappa_\varphi + \bar{a}\bar{p}\frac{\bar{c}}{\underline{r}}\kappa_\chi. & (3.10) \end{aligned}$$

Also define the following:

$$\begin{aligned} \kappa_{nonl} &:= \frac{\bar{\kappa}}{\underline{p}}\left(2\left(\bar{a} + \bar{a}\bar{p}\frac{\bar{c}^2}{\underline{r}}\right) + \bar{\kappa}\bar{\epsilon}\right), & (3.11)\\ \kappa_{noise} &:= \frac{S}{\underline{p}} + \frac{\bar{a}^2\bar{c}^2\bar{p}^2}{\underline{p}\underline{r}^2}, & (3.12)\\ \lambda &:= 1 - 1/\left(1 + \frac{\underline{q}}{\bar{p}\bar{a}^2(1+\bar{p}\bar{c}^2/\underline{r})^2}\right). & (3.13) \end{aligned}$$

Then

$$\epsilon = \min\left(\bar{\epsilon}, \frac{\lambda}{2\bar{p}\kappa_{nonl}}\right) \tag{3.14}$$

and

$$\delta = \frac{\lambda\epsilon^2}{2\bar{p}\kappa_{noise}}. \tag{3.15}$$

3.2 The Extended Kalman Filter Applied to the Target Tracking Problem

To apply the extended Kalman filter to this problem we obtain a linear approximation for the state equation. We approximate the non-linearity in the driving term as a time-varying linear function (that is $A_k = \bar{A}$ and $B_k = G(\bar{X}_{k|k-1})$ where $\bar{X}_{k|k-1}$ is the one-step-ahead prediction of X_k). The measurement equation (2.3) is non-linear in the state and linearisation at $\bar{X}_{k|k-1}$ gives

$$
\begin{aligned}
C_k &= \left.\frac{\partial c_k(X)}{\partial X}\right|_{X=\bar{X}_{k|k-1}} \\
&= \begin{bmatrix}
\hat{x}_{k|k-1}/\hat{R}_{k|k-1} & \hat{y}_{k|k-1}/\hat{R}_{k|k-1} & 0 & 0 \\
-\hat{y}_{k|k-1}/\hat{R}^2_{k|k-1} & \hat{x}_{k|k-1}/\hat{R}^2_{k|k-1} & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1/(1+\alpha^2) & -\alpha/(1+\alpha^2) \\
0 & 0 & \alpha/(1+\alpha^2) & 1/(1+\alpha^2) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\end{aligned}
\tag{3.16}
$$

where $\hat{R}_{k|k-1} = \sqrt{\bar{x}^2_{k|k-1} + \bar{y}^2_{k|k-1}}$.

The extended Kalman filter can now be implemented using the recursions (3.3) stated above.

3.3 Stability of the Extended Kalman Filter for Typical Configurations

To demonstrate how the extended Kalman filter stability results can be applied to this target tracking problem consider the typical configuration given in Section 2. Because of linear state equations the conditions for stability stated in Theorem 1 simplify to

$$\epsilon = \min\left(\epsilon_\chi, \frac{p\underline{r}\lambda}{2\bar{p}^2\kappa_\chi}\left(2(1+\frac{\bar{p}}{\underline{r}}) + \frac{\bar{p}\epsilon_\chi\kappa_\chi}{\underline{r}}\right)^{-1}\right) \tag{3.17}$$

where we have used $\bar{c} = 1$, $\bar{a} = 1$, that ϵ_φ is unbounded and $\kappa_\varphi = 0$. Note that Theorem 1 does not require bounds on R^*_k and Q^*_k in this target tracking problem because the state equation is linear.

A quick examination of (3.17) demonstrates that the stability of the EKF can always be ensured by setting $\underline{r}$ large enough. However, stability ensured by dramatically increasing R is at the cost of performance.

To determine whether stability of the extended Kalman filter can be ensured for a particular initial error value we tested values of ϵ_χ in (3.17). We investigated stability against initial errors in the y position coordinate (assuming no error in x_0). Figure 3 shows the values of ϵ achieved for various values of ϵ_χ (note that this figure shows only the stability at the initial time instant and the stability of the filter at later time instants needs to be tested separately). From Figure 3, stability of the EKF can be guaranteed for initial errors in y less that 180 m. Using (3.17) it can be shown that when y_0 is known, errors in x_0 do not cause the EKF to diverge.

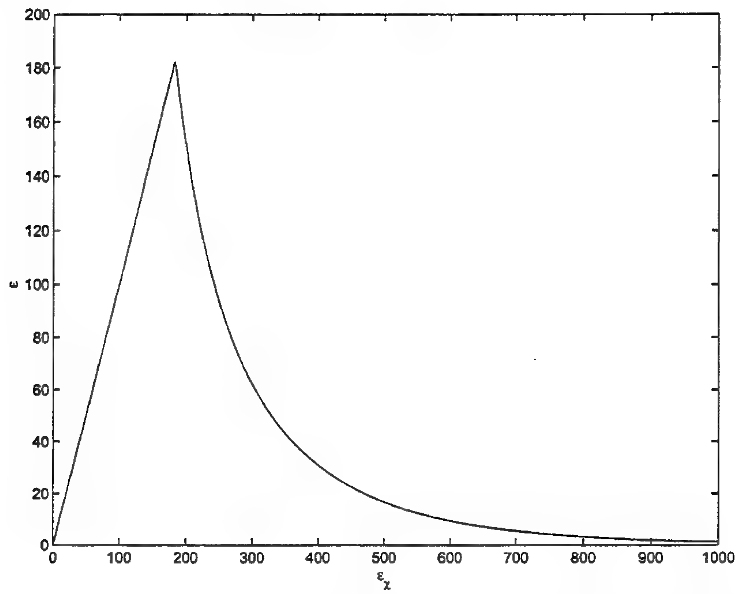

Figure 3 (U): The initial errors in y_0 for which the EKF is guaranteed to converge.

In Section 5 simulation results demonstrate that the EKF can converge from initial errors of 500 m in both axes. These simulations demonstrates that although useful, the bounds produced by (3.17) are conservative.

4 Implementation Issues

4.1 Measurement Noise Covariance Matrix

The stochastic stability results presented above suggest that convergence from a larger initialisation set can be guaranteed if the measurement noise covariance matrix is made larger than the actual noise covariance. This makes sense particularly when the uncertainty in the initial estimate is large.

The state equation and measurement equations linearisations are valid only for small state errors and it makes sense to increase the process and measurement noise covariance matrices (for a fixed $\bar{P}_{0|0}$) when the initialisations are known to be poor. Increasing these covariance matrices results in an EKF that can allow for larger mismatch between output predictions and actual measurements received. But, to obtain good filtering performance, the size of covariance matrices should only be increased during the initialisation period.

In this section we proposed one methodology for modifying the measurement noise matrix as

$$\bar{R}_k = R + C_k \bar{P}_{k|k-1} C_k' \tag{4.1}$$

where R is the covariance associated with $n_k(X_k)$ and $\bar{R}_k$ is the modified covariance to be used in the EKF.

Increasing the measurement noise covariance matrix in this way is an *ad hoc* modification and needs to be checked in simulation studies. Simulations presented in the next section show that this modification improves the convergence properties of the EKF.

5 Simulation Studies

Simulation studies of the extended Kalman filter were performed using Matlab^{TM}. Both the effect of initial errors and the effect of engagement configuration on performance of the EKF are examined. The simulations were performed with a sampling period of 0.001 s. In the below simulations the velocities of the interceptor and target are 1000 ms^{-1} and 660 ms^{-1} respectively.

We are interesting in the error performance of the EKF with respect to position, velocity, the perpendicular vector, and acceleration. However, the estimation performance of the

filter with respect to the all the state variables is closely coupled to the position error performance. Hence, most of the results presented in this section show only the position error performance. Similar estimation performance was obtained for the other state quantities.

5.1 The Typical Engagement

Consider the engagement shown in Figure 1. Assume that the target's position is measured indirectly via range and bearing measurements (with range noise variance of $0.25R_k\ m^2$, where R_k is the range at time k, and the angle noise variance of 0.25 rad^2) and assume that the target is non-accelerating. Assume that the initial estimate of the target position is $(5500, 500)\ m$ and that velocity errors of 5 ms^{-1} in both x and y directions are present. The EKF is then used to estimate the state of the target.

Figure 4 shows a plot of both the target and interceptor trajectories as well as the interceptor's estimate of the target's position. The initial position error quickly reduces and after 4.39 s, when the interceptor and target are 71m apart (which is the closest distance achieved), the error in the estimated target position is 0.67m.

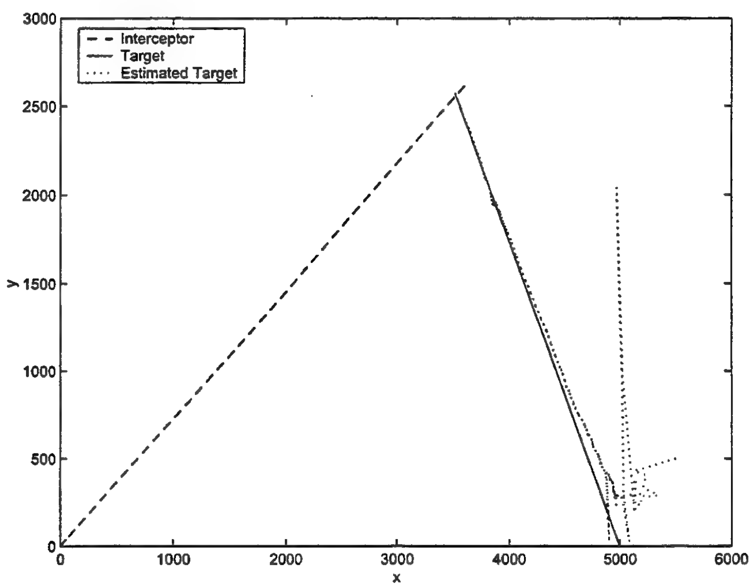

Figure 4 (U): Estimation of Target Position.

The velocity estimation errors are shown in Figure 5 and the perpendicular velocity esti-

mation errors are shown in Figure 6. These two figures demonstrate the non-linear nature of this filtering problem. Changes in the engagement configuration, over time, change the observability of various states. It is difficult to draw definite conclusions about the performance of the filter from the velocity estimation errors without considering the position estimation errors and the various covariance matrices. Because of this difficulty, position estimation errors are easier to use as a basis for comparison (rather that velocity estimation errors).

In the next section a more detailed examination of the filter's performance is given.

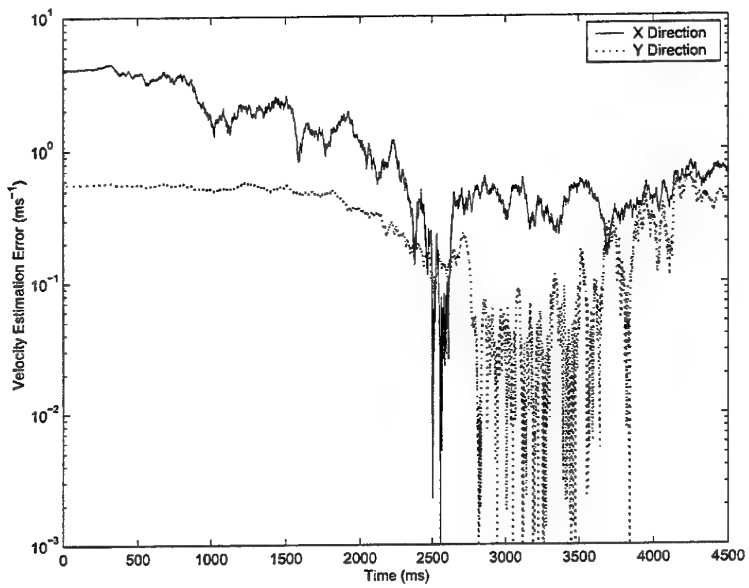

Figure 5 (U): Estimation of Target Velocity Vector.

5.1.1 Effect of Errors in the Initial Position Estimate

In this simulation study the effect of initial errors (in position) on the performance of the EKF is examined.

Consider the first 4.5 s of the engagement shown in Figure 1. Observations were generated with measurement noise variances of $0.01R_k\ m^2$ and $0.01\ rad^2$ for the range and bearing measurements respectively.

To evaluate the sensitivity of the EKF to initialisation, the EKF was applied to the gener-

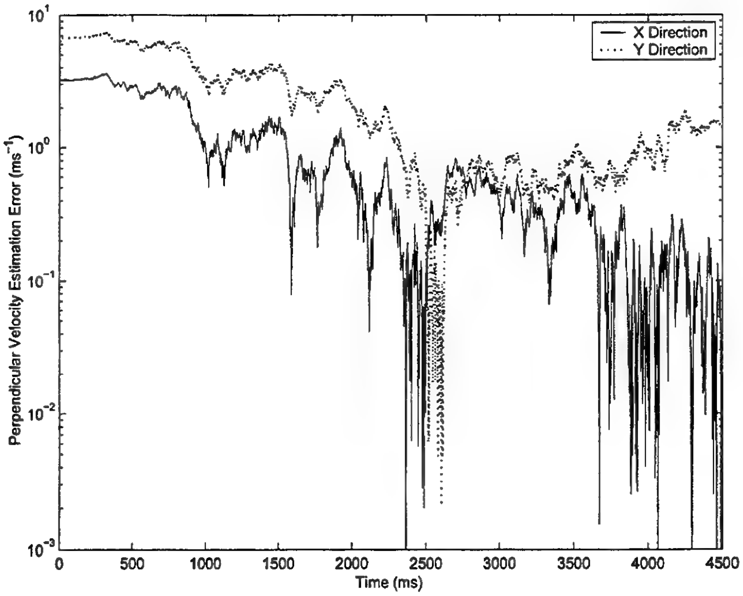

Figure 6 (U): Estimation of Target Perpendicular Velocity Vector.

ated observations when initialised with x and y position errors in the range $(-2500, 2500)$ m while initial velocity estimates errors were fixed at 5 ms^{-1}.

For each initialisation condition, the performance of the EKF was measured as the time average of the position estimation error over the last 20% of the engagement time. Ten runs for each initialisation error were performed and average error performance over these ten runs is shown in Figure 7.

Figure 7 shows two surfaces representing the performance of the EKF with and without the covariance matrix modification described in Section 4. The surfaces show the average final position error for different initialisation errors. The flat surface is the average error performance of the EKF with the modified covariance matrix and the peaked surface is the average error performance of the standard EKF.

From these curves it is clear that initialisation errors significantly influence the performance of the standard EKF. The influence of initialisation errors can be reduced by increasing the size of the noise covariance matrix (particularly during the initial stages) to ensure stability of the filter. The performance of the EKF with modified covariance matrix is superior to the standard EKF except when the initial errors are small.

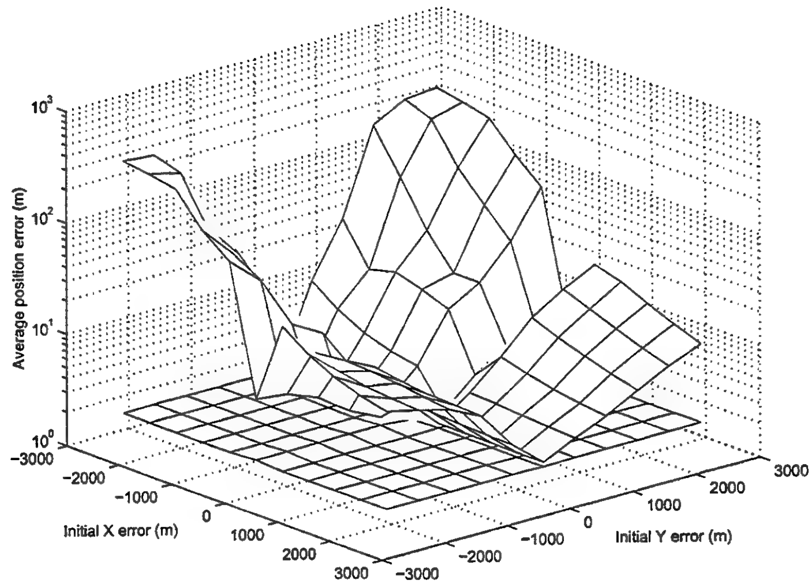

Figure 7 (U): Error performance for various initialisations. The flat curve is the EKF with a modified covariance matrix and the peaked curve is the standard EKF.

5.2 Effect of Configuration on Estimation

The engagements shown in Figure 8 were simulated to examine the effect of engagement geometry on the performance of the EKF (without the modified covariance matrix). Eleven separate configurations where considered; those corresponding to the interceptor approaching the target from angles between $-90°$ to $90°$ in $15°$ steps. For each configuration the engagement was simulated for 4.5 s with the simulation ending with a final separation of 100 m between interceptor and target.

For each configuration, the filter was initialised with 36 different initial errors in position (placed every $10°$ around a circle of radius 500 m centered at the initial target position). The initial error in both the velocity and acceleration was assumed to be zero. Using the same noise sequences, w_k and v_k, data was generated each of the 11 configurations and then the EKF was applied to this data using each of the 36 possible initialisations. This process was repeated ten times with ten noise sequences (ie. the EKF was applied 3960 times).

For each configuration, the average error over the last 20% of the engagement was averaged

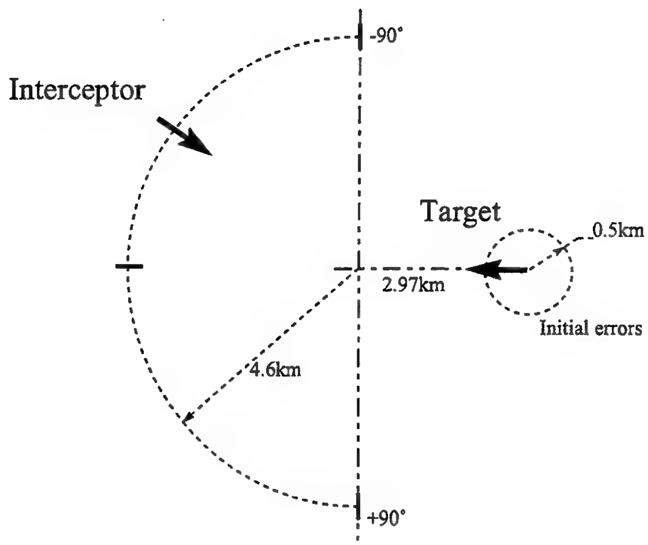

Figure 8 (U): Various engagement configurations.

over all 36 initialisations and then averaged over all 10 runs. The average x, y and total position error are shown against the configuration angle in Figure 9.

Figure 9 shows the effect of engagement geometry on the error performance of the filter. The average error in the x (or y) coordinate of position estimate increases (or decreases) with angle away from 0°. Overall, the total position error decreases with angle away from 0°.

The engagement geometry also influences the ability of the filter to estimate the velocity vectors (including the perpendicular vector). In fact, estimating various components of the velocity vectors is very difficult in some configurations. Simulation results examining velocity estimates have not been presented because other effects (including errors in position estimates) are highly coupled to these velocity errors. The effect on velocity estimates is best understood by considering the flow through effect of position errors together with the observability grammian.

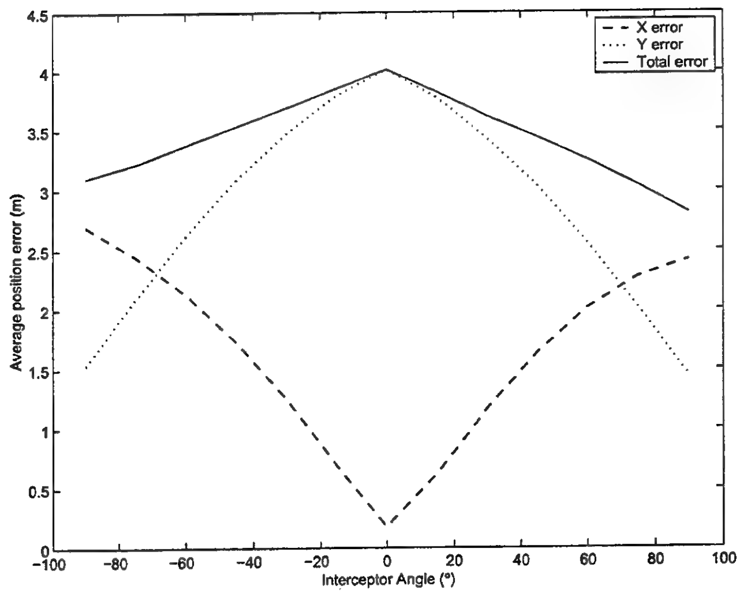

Figure 9 (U): Error performance for various engagement geometries.

6 Conclusions

This paper examined the used of the extended Kalman filter for estimating the target state. The target state estimation problem considered in this paper is different from the usual estimation problem because in this case it is necessary to estimate information in addition (in particular the perpendicular vector) to the usual cartesian position and velocity. An extended Kalman filtering solution for this problem is presented and simulation studies are performed that demonstrate the stability and error performance of filter.

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19. ABSTRACT

This paper examines the use of the extended Kalman filter for estimating various quantities in typical interceptor-target engagements. The extended Kalman filter is used to estimate the relative position of the target, the relative velocity of the target and the vector perpendicular to the target velocity. The target is assumed to be non accelerating. The target is observed via range and bearing measurements and it is assumed that the interceptor's own velocities are known.

The performance and stability of the extended Kalman filter are examined under a variety of initialisation errors, engagement configurations, and measurement noise variances. Simulation studies demonstrate that the required quantities can be estimated but that the performance of the filter is dependent on the configuration of the engagement.